

THE USE OF QUANTUM GROUPS IN NUCLEAR STRUCTURE PROBLEMS

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ABSTRACT

Various applications of quantum algebraic techniques in nuclear structure physics, such as the $\text{su}_q(2)$ rotator model and its extensions, the use of deformed bosons in the description of pairing correlations, and the construction of deformed exactly soluble models (Interacting Boson Model, Moszkowski model) are briefly reviewed. Emphasis is put in the study of the symmetries of the anisotropic quantum harmonic oscillator with rational ratios of frequencies, which underly the structure of superdeformed and hyperdeformed nuclei, the Bloch–Brink α -cluster model and possibly the shell structure in deformed atomic clusters.

1. Introduction

Quantum algebras^{1,2} (also called quantum groups) are deformed versions of the usual Lie algebras, to which they reduce when the deformation parameter q is set equal to unity. Their use in physics became popular with the introduction^{3–5} of the q -deformed harmonic oscillator as a tool for providing a boson realization of the quantum algebra $\text{su}_q(2)$, although similar mathematical structures had already been known^{6,7}. Initially used for solving the quantum Yang–Baxter equation, quantum algebras have subsequently found applications in several branches of physics, as, for example, in the description of spin chains, squeezed states, rotational and vibrational nuclear and molecular spectra, and in conformal field theories. By now several kinds of generalized deformed oscillators^{8–12} and generalized deformed $\text{su}(2)$ algebras^{13–19} have been introduced.

Here we shall confine ourselves to applications of quantum algebras in nuclear structure physics. A brief description will be given of the $\text{su}_q(2)$ rotator model^{20–25} and its extensions^{13,26}, of the use of deformed oscillators in the description of pairing correlations^{27–29}, and of the formulation of deformed exactly soluble models (Interacting Boson Model³⁰, Moszkowski model^{31–33}). The purpose of this short review is to provide the reader with references for further reading. Subsequently, the symmetries of the anisotropic quantum harmonic oscillator with rational ratios of frequencies will be considered in more detail, since they are of current interest^{34,35} in connection

with superdeformed and hyperdeformed nuclei ^{36,37}, α -cluster configurations in light nuclei ^{38–40}, and possibly with deformed atomic clusters ^{41,42}.

2. The $\mathbf{su}_q(2)$ rotator model

The first application of quantum algebras in nuclear physics was the use of the deformed algebra $\mathbf{su}_q(2)$ for the description of the rotational spectra of deformed ^{20,21} and superdeformed ²² nuclei. The same technique has been used for the description of rotational spectra of diatomic molecules ²³. The Hamiltonian of the q -deformed rotator is proportional to the second order Casimir operator of the $\mathbf{su}_q(2)$ algebra. Its Taylor expansion contains powers of $J(J + 1)$ (where J is the angular momentum), being similar to the expansion provided by the Variable Moment of Inertia (VMI) model. Furthermore, the deformation parameter τ (with $q = e^{i\tau}$) has been found to correspond to the softness parameter of the VMI model ²¹.

$B(E2)$ transition probabilities have also been described in this framework ²⁴. In this case the q -deformed Clebsch–Gordan coefficients are used instead of the normal ones. (It should be noticed that the q -deformed angular momentum theory has already been much developed ²⁴.) The model predicts an increase of the $B(E2)$ values with angular momentum, while the rigid rotator model predicts saturation. Some experimental results supporting this prediction already exist ²⁴.

3. Extensions of the $\mathbf{su}_q(2)$ model

The $\mathbf{su}_q(2)$ model has been successful in describing rotational nuclear spectra. For the description of vibrational and transitional nuclear spectra it has been found ²⁶ that $J(J + 1)$ has to be replaced by $J(J + c)$. The additional parameter c allows for the description of nuclear anharmonicities in a way similar to that of the Interacting Boson Model (IBM) ^{43,44} and the Generalized Variable Moment of Inertia (GVMI) model ⁴⁵.

Another generalization is based on the use of the deformed algebra $\mathbf{su}_\Phi(2)$ ¹³, which is characterized by a structure function Φ . The usual $\mathbf{su}(2)$ and $\mathbf{su}_q(2)$ algebras are obtained for specific choices of the structure function Φ . The $\mathbf{su}_\Phi(2)$ algebra has been constructed so that its representation theory resembles as much as possible the representation theory of the usual $\mathbf{su}(2)$ algebra. Using this technique one can construct, for example, a rotator having the same spectrum as the one given by the Holmberg–Lipas formula ⁴⁶. In addition to the generalized deformed $\mathbf{su}(2)$ algebra, generalized deformed oscillators ^{8–12} have also been introduced and found useful in many physical applications.

4. Pairing correlations

It has been found ²⁷ that correlated fermion pairs coupled to zero angular momentum in a single- j shell behave approximately as suitably defined q -deformed bosons.

After performing the same boson mapping to a simple pairing Hamiltonian, one sees that the pairing energies are also correctly reproduced up to the same order. The deformation parameter used ($\tau = \ln q$) is found to be inversely proportional to the size of the shell, thus serving as a small parameter.

The above mentioned system of correlated fermion pairs can be described exactly by suitably defined generalized deformed bosons ²⁸. Then both the commutation relations are satisfied exactly and the pairing energies are reproduced exactly. The spectrum of the appropriate generalized deformed oscillator corresponds, up to first order perturbation theory, to a harmonic oscillator with an x^4 perturbation.

5. *q*-deformed versions of nuclear models

A *q*-deformed version of a two dimensional toy Interacting Boson Model (IBM) has been developed ³⁰, mainly for testing the ways in which spectra and transition probabilities are influenced by the *q*-deformation. A *q*-deformed version of the full IBM is under development, while a *q*-deformed version of the vibron model, which uses the IBM techniques in the case of molecules, has already been developed ⁴⁷.

Furthermore a *q*-deformed version of the Moszkowski model has been developed ³¹ and RPA modes have been studied ³² in it. A *q*-deformed Moszkowski model with cranking has also been studied ³³ in the mean-field approximation. It has been seen that the residual interaction simulated by the *q*-deformation is felt more strongly by states with large J_z . The possibility of using *q*-deformation in assimilating temperature effects is under discussion.

6. Anisotropic quantum harmonic oscillator with rational ratios of frequencies

The symmetries of the 3-dimensional anisotropic quantum harmonic oscillator with rational ratios of frequencies (RHO) are of high current interest in nuclear physics, since they are the basic symmetries ^{34,35} underlying the structure of superdeformed and hyperdeformed nuclei ^{36,37}. The 2-dimensional RHO is also of interest, in connection with “pancake” nuclei, i.e. very oblate nuclei ³⁵. Cluster configurations in light nuclei can also be described in terms of RHO symmetries ^{38,39}, which underlie the geometrical structure of the Bloch–Brink α -cluster model ⁴⁰. The 3-dim RHO is also of interest for the interpretation of the observed shell structure in atomic clusters ⁴¹, especially after the realization ⁴² that large deformations can occur in such systems.

The two-dimensional ^{48–53} and three-dimensional ^{54–60} anisotropic harmonic oscillators have been the subject of several investigations, both at the classical and the quantum mechanical level. These oscillators are examples of superintegrable systems ⁶¹. The special cases with frequency ratios 1:2 ^{62,63} and 1:3 ⁶⁴ have also been considered. While at the classical level it is clear that the $\text{su}(N)$ or $\text{sp}(2N, R)$ algebras can be used for the description of the N -dimensional anisotropic oscillator, the situation

at the quantum level, even in the two-dimensional case, is not as simple.

In this section we are going to prove that a generalized deformed $u(2)$ algebra is the symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator, which is the oscillator describing the single-particle level spectrum of “pancake” nuclei, i.e. of triaxially deformed nuclei with $\omega_x \gg \omega_y, \omega_z$ ³⁵.

6.1. The deformed $u(2)$ algebra

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2} \left(p_x^2 + p_y^2 + \frac{x^2}{m^2} + \frac{y^2}{n^2} \right), \quad (1)$$

where m and n are two natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m, n) = 1$.

We define the creation and annihilation operators⁴⁸

$$\begin{aligned} a^\dagger &= \frac{x/m - ip_x}{\sqrt{2}}, & a &= \frac{x/m + ip_x}{\sqrt{2}}, \\ b^\dagger &= \frac{y/n - ip_y}{\sqrt{2}}, & b &= \frac{y/n + ip_y}{\sqrt{2}}. \end{aligned} \quad (2)$$

These operators satisfy the commutation relations:

$$[a, a^\dagger] = \frac{1}{m}, \quad [b, b^\dagger] = \frac{1}{n}, \quad \text{other commutators} = 0. \quad (3)$$

Using Eqs (2) and (3) we can prove by induction that:

$$\begin{aligned} [a, (a^\dagger)^p] &= \frac{p}{m} (a^\dagger)^{p-1}, & [b, (b^\dagger)^p] &= \frac{p}{n} (b^\dagger)^{p-1}, \\ [a^\dagger, (a)^p] &= -\frac{p}{m} (a)^{p-1}, & [b^\dagger, (b)^p] &= -\frac{p}{n} (b)^{p-1}. \end{aligned}$$

Defining

$$U = \frac{1}{2} \{a, a^\dagger\}, \quad W = \frac{1}{2} \{b, b^\dagger\},$$

one can easily prove that:

$$\begin{aligned} [U, (a^\dagger)^p] &= \frac{p}{m} (a^\dagger)^p, & [W, (b^\dagger)^p] &= \frac{p}{n} (b^\dagger)^p, \\ [U, (a)^p] &= -\frac{p}{m} (a)^p, & [W, (b)^p] &= -\frac{p}{n} (b)^p. \end{aligned}$$

Using the above properties we can define the enveloping algebra generated by the operators:

$$\begin{aligned} S_+ &= (a^\dagger)^m (b)^n, & S_- &= (a)^m (b^\dagger)^n, \\ S_0 &= \frac{1}{2} (U - W), & H &= U + W. \end{aligned} \quad (4)$$

These generators satisfy the following relations:

$$[S_0, S_{\pm}] = \pm S_{\pm}, \quad [H, S_i] = 0, \quad \text{for } i = 0, \pm, \quad (5)$$

and

$$\begin{aligned} S_+ S_- &= \prod_{k=1}^m \left(U - \frac{2k-1}{2m} \right) \prod_{\ell=1}^n \left(W + \frac{2\ell-1}{2n} \right), \\ S_- S_+ &= \prod_{k=1}^m \left(U + \frac{2k-1}{2m} \right) \prod_{\ell=1}^n \left(W - \frac{2\ell-1}{2n} \right). \end{aligned}$$

The fact that the operators S_i , $i = 0, \pm$ are integrals of motion has been already realized in ⁴⁸.

The above relations mean that the harmonic oscillator of Eq. (1) is described by the enveloping algebra of the generalization of the $u(2)$ algebra formed by the generators S_0 , S_+ , S_- and H , satisfying the commutation relations of Eq. (5) and

$$\begin{aligned} [S_-, S_+] &= F_{m,n}(H, S_0 + 1) - F_{m,n}(H, S_0), \\ \text{where } F_{m,n}(H, S_0) &= \prod_{k=1}^m \left(H/2 + S_0 - \frac{2k-1}{2m} \right) \prod_{\ell=1}^n \left(H/2 - S_0 + \frac{2\ell-1}{2n} \right). \end{aligned} \quad (6)$$

In the case of $m = 1$, $n = 1$ this algebra is the usual $u(2)$ algebra, and the operators S_0 , S_{\pm} satisfy the commutation relations of the ordinary $u(2)$ algebra, since in this case one easily finds that

$$[S_-, S_+] = -2S_0.$$

In the rest of the cases, the algebra is a deformed version of $u(2)$, in which the commutator $[S_-, S_+]$ is a polynomial of S_0 of order $m + n - 1$. In the case with $m = 1$, $n = 2$ one has

$$[S_-, S_+] = 3S_0^2 - HS_0 - \frac{H^2}{4} + \frac{3}{16},$$

i.e. a polynomial quadratic in S_0 occurs, while in the case of $m = 1$, $n = 3$ one finds

$$[S_-, S_+] = -4S_0^3 + 3HS_0^2 - \frac{7}{9}S_0 - \frac{H^3}{4} + \frac{H}{4},$$

i.e. a polynomial cubic in S_0 is obtained.

6.2. The representations

The finite dimensional representation modules of this algebra can be found using the concept of the generalized deformed oscillator ⁸, in a method similar to the one used in Ref. 65 for the study of quantum superintegrable systems. The operators:

$$\mathcal{A}^\dagger = S_+, \quad \mathcal{A} = S_-, \quad \mathcal{N} = S_0 - u, \quad u = \text{constant}, \quad (7)$$

where u is a constant to be determined, are the generators of a deformed oscillator algebra:

$$[\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad \mathcal{A}^\dagger \mathcal{A} = \Phi(H, \mathcal{N}), \quad \mathcal{A} \mathcal{A}^\dagger = \Phi(H, \mathcal{N} + 1).$$

The structure function Φ of this algebra is determined by the function $F_{m,n}$ in Eq. (6):

$$\begin{aligned} \Phi(H, \mathcal{N}) &= F_{m,n}(H, \mathcal{N} + u) = \\ &= \prod_{k=1}^m \left(H/2 + \mathcal{N} + u - \frac{2k-1}{2m} \right) \prod_{\ell=1}^n \left(H/2 - \mathcal{N} - u + \frac{2\ell-1}{2n} \right). \end{aligned} \quad (8)$$

The deformed oscillator corresponding to the structure function of Eq. (8) has an energy dependent Fock space of dimension $N + 1$ if

$$\Phi(E, 0) = 0, \quad \Phi(E, N + 1) = 0, \quad \Phi(E, k) > 0, \quad \text{for } k = 1, 2, \dots, N. \quad (9)$$

The Fock space is defined by:

$$H|E, k\rangle = E|E, k\rangle, \quad \mathcal{N}|E, k\rangle = k|E, k\rangle, \quad a|E, 0\rangle = 0, \quad (10)$$

$$\mathcal{A}^\dagger|E, k\rangle = \sqrt{\Phi(E, k + 1)}|E, k + 1\rangle, \quad \mathcal{A}|E, k\rangle = \sqrt{\Phi(E, k)}|E, k - 1\rangle. \quad (11)$$

The basis of the Fock space is given by:

$$|E, k\rangle = \frac{1}{\sqrt{[k]!}} (\mathcal{A}^\dagger)^k |E, 0\rangle, \quad k = 0, 1, \dots, N,$$

where the “factorial” $[k]!$ is defined by the recurrence relation:

$$[0]! = 1, \quad [k]! = \Phi(E, k)[k - 1]!. \quad .$$

Using the Fock basis we can find the matrix representation of the deformed oscillator and then the matrix representation of the algebra of Eqs (5), (6). The solution of Eqs (9) implies the following pairs of permitted values for the energy eigenvalue E and the constant u :

$$E = N + \frac{2p - 1}{2m} + \frac{2q - 1}{2n}, \quad (12)$$

where $p = 1, 2, \dots, m$, $q = 1, 2, \dots, n$, and

$$u = \frac{1}{2} \left(\frac{2p - 1}{2m} - \frac{2q - 1}{2n} - N \right),$$

the corresponding structure function being given by:

$$\begin{aligned} \Phi(E, x) &= \Phi_{(p,q)}^N(x) = \\ &= \prod_{k=1}^m \left(x + \frac{2p - 1}{2m} - \frac{2k - 1}{2m} \right) \prod_{\ell=1}^n \left(N - x + \frac{2q - 1}{2n} + \frac{2\ell - 1}{2n} \right) \\ &= \frac{1}{m^m n^n} \frac{\Gamma(mx + p)}{\Gamma(mx + p - m)} \frac{\Gamma((N - x)n + q + n)}{\Gamma((N - x)n + q)}. \end{aligned} \quad (13)$$

In all these equations one has $N = 0, 1, 2, \dots$, while the dimensionality of the representation is given by $N + 1$. Eq. (12) means that there are $m \cdot n$ energy eigenvalues corresponding to each N value, each eigenvalue having degeneracy $N + 1$. (Later we shall see that the degenerate states corresponding to the same eigenvalue can be labelled by an “angular momentum”.)

It is useful to show at this point that a few special cases are in agreement with results already existing in the literature.

i) In the case $m = 1, n = 1$ Eq. (13) gives

$$\Phi(E, x) = x(N + 1 - x),$$

while Eq. (12) gives

$$E = N + 1,$$

in agreement with Sec. IV.A of Ref. 65.

ii) In the case $m = 1, n = 2$ one obtains for $q = 2$

$$\Phi(E, x) = x(N + 1 - x) \left(N + \frac{3}{2} - x \right), \quad E = N + \frac{5}{4},$$

while for $q = 1$ one has

$$\Phi(E, x) = x(N + 1 - x) \left(N + \frac{1}{2} - x \right), \quad E = N + \frac{3}{4}.$$

These are in agreement with the results obtained in Sec. IV.F of Ref. 65 for the Holt potential (for $\delta = 0$).

iii) In the case $m = 1, n = 3$ one has for $q = 1$

$$\Phi(E, x) = x(N + 1 - x) \left(N + \frac{1}{3} - x \right) \left(N + \frac{2}{3} - x \right), \quad E = N + \frac{2}{3},$$

while for $q = 2$ one obtains

$$\Phi(E, x) = x(N + 1 - x) \left(N + \frac{2}{3} - x \right) \left(N + \frac{4}{3} - x \right), \quad E = N + 1,$$

and for $q = 3$ one gets

$$\Phi(E, x) = x(N + 1 - x) \left(N + \frac{4}{3} - x \right) \left(N + \frac{5}{3} - x \right), \quad E = N + \frac{4}{3}.$$

These are in agreement with the results obtained in Sec. IV.D of Ref. 65 for the Fokas–Lagerstrom potential.

In all of the above cases we remark that the structure function has the form

$$\Phi(x) = x(N + 1 - x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \dots),$$

which corresponds to a generalized deformed parafermionic algebra ⁶⁶ of order N , if $\lambda, \mu, \nu, \rho, \sigma, \dots$, are real constants satisfying the conditions

$$\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \dots > 0, \quad x \in \{1, 2, \dots, N\}.$$

These conditions are indeed satisfied in all cases. It is easy to see that the obtained algebra corresponds to this of the generalized parafermionic oscillator in all cases with frequency ratios $1 : n$.

The energy formula can be corroborated by using the corresponding Schrödinger equation. For the Hamiltonian of Eq. (1) the eigenvalues of the Schrödinger equation are given by:

$$E = \frac{1}{m} \left(n_x + \frac{1}{2} \right) + \frac{1}{n} \left(n_y + \frac{1}{2} \right), \quad (14)$$

where $n_x = 0, 1, \dots$ and $n_y = 0, 1, \dots$. Comparing Eqs (12) and (14) one concludes that:

$$N = [n_x/m] + [n_y/n],$$

where $[x]$ is the integer part of the number x , and

$$p = \text{mod}(n_x, m) + 1, \quad q = \text{mod}(n_y, n) + 1.$$

The eigenvectors of the Hamiltonian can be parametrized by the dimensionality of the representation N , the numbers p, q , and the number $k = 0, 1, \dots, N$. k can be identified as $[n_x/m]$. One then has:

$$H \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k \right\rangle = \left(N + \frac{2p-1}{2m} + \frac{2q-1}{2n} \right) \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k \right\rangle, \quad (15)$$

$$S_0 \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k \right\rangle = \left(k + \frac{1}{2} \left(\frac{2p-1}{2m} - \frac{2q-1}{2n} - N \right) \right) \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k \right\rangle, \quad (16)$$

$$S_+ \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k \right\rangle = \sqrt{\Phi_{(p,q)}^N(k+1)} \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k+1 \right\rangle, \quad (17)$$

$$S_- \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k \right\rangle = \sqrt{\Phi_{(p,q)}^N(k)} \left| \begin{smallmatrix} N \\ (p, q) \end{smallmatrix} , k-1 \right\rangle. \quad (18)$$

6.3. The “angular momentum” quantum number

It is worth noticing that the operators S_0, S_{\pm} do not correspond to a generalization of the angular momentum, S_0 being the operator corresponding to the Fradkin operator $S_{xx} - S_{yy}$ ^{67,68}. The corresponding “angular momentum” is defined by:

$$L_0 = -i(S_+ - S_-). \quad (19)$$

The “angular momentum” operator commutes with the Hamiltonian:

$$[H, L_0] = 0.$$

Let $|\ell\rangle$ be the eigenvector of the operator L_0 corresponding to the eigenvalue ℓ . The general form of this eigenvector can be given by:

$$|\ell\rangle = \sum_{k=0}^N \frac{i^k c_k}{\sqrt{[k]!}} \left| \begin{array}{c} N \\ (p, q) \end{array} , k \right\rangle. \quad (20)$$

In order to find the eigenvalues of L and the coefficients c_k we use the Lanczos algorithm⁶⁹, as formulated in⁷⁰. From Eqs (17) and (18) we find

$$\begin{aligned} L_0|\ell\rangle &= \ell|\ell\rangle = \ell \sum_{k=0}^N \frac{i^k c_k}{\sqrt{[k]!}} \left| \begin{array}{c} N \\ (p, q) \end{array} , k \right\rangle = \\ &= \frac{1}{i} \sum_{k=0}^{N-1} \frac{i^k c_k \sqrt{\Phi_{(p,q)}^N(k+1)}}{\sqrt{[k]!}} \left| \begin{array}{c} N \\ (p, q) \end{array} , k+1 \right\rangle - \frac{1}{i} \sum_{k=1}^N \frac{i^k c_k \sqrt{\Phi_{(p,q)}^N(k)}}{\sqrt{[k]!}} \left| \begin{array}{c} N \\ (p, q) \end{array} , k-1 \right\rangle \end{aligned}$$

From this equation we find that:

$$c_k = (-1)^k 2^{-k/2} H_k(\ell/\sqrt{2})/\mathcal{N}, \quad \mathcal{N}^2 = \sum_{n=0}^N 2^{-n} H_n^2(\ell/\sqrt{2})$$

where the function $H_k(x)$ is a generalization of the “Hermite” polynomials (see also^{71,72}), satisfying the recurrence relations:

$$\begin{aligned} H_{-1}(x) &= 0, \quad H_0(x) = 1, \\ H_{k+1}(x) &= 2xH_k(x) - 2\Phi_{(p,q)}^N(k)H_{k-1}(x), \end{aligned}$$

and the “angular momentum” eigenvalues ℓ are the roots of the polynomial equation:

$$H_{N+1}(\ell/\sqrt{2}) = 0. \quad (21)$$

Therefore for a given value of N there are $N+1$ “angular momentum” eigenvalues ℓ , symmetric around zero (i.e. if ℓ is an “angular momentum” eigenvalue, then $-\ell$ is also an “angular momentum” eigenvalue). In the case of the symmetric harmonic oscillator ($m/n = 1/1$) these eigenvalues are uniformly distributed and differ by 2. In the general case the “angular momentum” eigenvalues are non-uniformly distributed. For small values of N analytical formulae for the “angular momentum” eigenvalues can be found⁷¹. Remember that to each value of N correspond $m \cdot n$ energy levels, each with degeneracy $N+1$.

In order to have a formalism corresponding to the one of the isotropic oscillator, let us introduce for every N and (p, q) an ordering of the “angular momentum” eigenvalues

$$\ell_m^{L,(p,q)}, \quad \text{where } L = N \quad \text{and} \quad m = -L, -L+2, \dots, L-2, L,$$

by assuming that:

$$\ell_m^{L,(p,q)} \leq \ell_n^{L,(p,q)} \quad \text{if } m < n,$$

the corresponding eigenstate being given by:

$$|L, m; (p, q)\rangle = \sum_{k=0}^N \frac{(-i)^k H_k(\ell_m^{L,(p,q)} / \sqrt{2})}{\mathcal{N} \sqrt{2^{k/2} [k]!}} \left| \begin{array}{c} N \\ (p, q) \end{array} , k \right\rangle. \quad (22)$$

The above vector elements constitute the analogue corresponding to the basis of “spherical harmonic” functions of the usual oscillator.

6.4. Examples

For illustrative purposes, let us discuss a couple of examples in more detail.

i) In the case $m = 1, n = 1$ the only (p, q) value allowed is $(1, 1)$. In the Cartesian notation $|n_x n_y\rangle$ the lowest energy corresponds to the state $|00\rangle$, the next energy level corresponds to the two states $|10\rangle$ and $|01\rangle$, while the next energy corresponds to the three states $|20\rangle$, $|11\rangle$, $|02\rangle$. In the notation used in Eqs (15)-(18) these states can be written as

$$\begin{aligned} N(p, q) = 0(1, 1), \quad E = 1 \quad \rightarrow \quad |00\rangle &= \left| \begin{array}{c} 0 \\ (1, 1) \end{array} , 0 \right\rangle, \\ N(p, q) = 1(1, 1), \quad E = 2 \quad \rightarrow \quad &\left\{ \begin{array}{l} |01\rangle = \left| \begin{array}{c} 1 \\ (1, 1) \end{array} , 0 \right\rangle \\ |10\rangle = \left| \begin{array}{c} 1 \\ (1, 1) \end{array} , 1 \right\rangle, \end{array} \right. \\ N(p, q) = 2(1, 1), \quad E = 3 \quad \rightarrow \quad &\left\{ \begin{array}{l} |02\rangle = \left| \begin{array}{c} 2 \\ (1, 1) \end{array} , 0 \right\rangle \\ |11\rangle = \left| \begin{array}{c} 2 \\ (1, 1) \end{array} , 1 \right\rangle \\ |20\rangle = \left| \begin{array}{c} 2 \\ (1, 1) \end{array} , 2 \right\rangle. \end{array} \right. \end{aligned}$$

From this example it is clear that the irreducible representations (irreps) are characterized by the quantum numbers N and (p, q) , while k enumerates the degenerate states within each irrep. The lowest irrep, characterized by $N = 0$ and $(p, q) = (1, 1)$, has dimension $d = 1$, while the next irrep has $N = 1$, $(p, q) = (1, 1)$ and $d = 2$ and the third one $N = 2$, $(p, q) = (1, 1)$ and $d = 3$.

Using the “angular momentum” eigenvalues defined by Eqs (21) and (22) we find:

$$\begin{aligned} E = 1 \quad \rightarrow \quad |0, 0; (1, 1)\rangle &= |0, 0\rangle, \quad \leftarrow \quad \ell_0^{0,(1,1)} = 0, \\ E = 2 \quad \rightarrow \quad \left\{ \begin{array}{l} |1, -1; (1, 1)\rangle = \frac{1}{\sqrt{2}} (|01\rangle + i|1, 0\rangle) \quad \leftarrow \quad \ell_{(-1)}^{1,(1,1)} = -1 \\ |1, +1; (1, 1)\rangle = \frac{1}{\sqrt{2}} (|01\rangle - i|1, 0\rangle) \quad \leftarrow \quad \ell_{(+1)}^{1,(1,1)} = 1, \end{array} \right. \end{aligned}$$

$$E = 3 \rightarrow \begin{cases} |2, -2; (1, 1)\rangle = \frac{1}{2}|0, 2\rangle + \frac{i}{\sqrt{2}}|1, 1\rangle - \frac{1}{2}|2, 0\rangle & \leftarrow \ell_{(-2)}^{2,(1,1)} = -2 \\ |2, 0; (1, 1)\rangle = \frac{1}{\sqrt{2}}|0, 2\rangle + \frac{1}{\sqrt{2}}|2, 0\rangle & \leftarrow \ell_{(0)}^{2,(1,1)} = 0 \\ |2, +2; (1, 1)\rangle = \frac{1}{2}|0, 2\rangle - \frac{i}{\sqrt{2}}|1, 1\rangle - \frac{1}{2}|2, 0\rangle & \leftarrow \ell_{(+2)}^{2,(1,1)} = +2. \end{cases}$$

We see therefore that the angular momentum eigenvalues l can be used, instead of k , for enumerating the degenerate states within each irrep.

ii) In the case $m = 1, n = 2$ the allowed values of (p, q) are $(1, 1)$ and $(1, 2)$. The lowest irrep is characterized by $N = 0, (p, q) = (1, 1)$, has dimension $d = 1$ and contains the state $|n_x, n_y\rangle = |00\rangle$, the next irrep is characterized by $N = 0, (p, q) = (1, 2)$, has $d = 1$ and contains the state $|01\rangle$, the third irrep has $N = 1, (p, q) = (1, 1), d = 2$ and contains the states $|10\rangle$ and $|02\rangle$, the fourth irrep has $N = 1, (p, q) = (1, 2), d = 2$ and contains the states $|11\rangle$ and $|03\rangle$, the fifth irrep has $N = 2, (p, q) = (1, 1), d = 3$ and contains the states $|20\rangle, |12\rangle$ and $|04\rangle$, the sixth irrep has $N = 2, (p, q) = (1, 2), d = 3$ and contains the states $|21\rangle, |13\rangle$ and $|05\rangle$. The states are listed in both notations below:

$$\begin{aligned} E = \frac{3}{4} \rightarrow |00\rangle &= \begin{vmatrix} 0 \\ (1, 1) \end{vmatrix}, & E = \frac{5}{4} \rightarrow |01\rangle &= \begin{vmatrix} 0 \\ (1, 2) \end{vmatrix}, \\ E = \frac{7}{4} \rightarrow \begin{cases} |02\rangle = \begin{vmatrix} 1 \\ (1, 1) \end{vmatrix}, \\ |10\rangle = \begin{vmatrix} 1 \\ (1, 1) \end{vmatrix}, \end{cases} & & E = \frac{9}{4} \rightarrow \begin{cases} |03\rangle = \begin{vmatrix} 1 \\ (1, 2) \end{vmatrix}, \\ |11\rangle = \begin{vmatrix} 1 \\ (1, 2) \end{vmatrix}, \end{cases}, \\ E = \frac{11}{4} \rightarrow \begin{cases} |04\rangle = \begin{vmatrix} 2 \\ (1, 1) \end{vmatrix}, \\ |12\rangle = \begin{vmatrix} 2 \\ (1, 1) \end{vmatrix}, \\ |20\rangle = \begin{vmatrix} 2 \\ (1, 1) \end{vmatrix}, \end{cases} & & E = \frac{13}{4} \rightarrow \begin{cases} |05\rangle = \begin{vmatrix} 2 \\ (1, 2) \end{vmatrix}, \\ |13\rangle = \begin{vmatrix} 2 \\ (1, 2) \end{vmatrix}, \\ |21\rangle = \begin{vmatrix} 2 \\ (1, 2) \end{vmatrix}. \end{cases} \end{aligned}$$

Using the “angular momentum” eigenvalues defined by Eqs (21) and (22) we find:

$$\begin{aligned} E = \frac{3}{4} \rightarrow |0, 0; (1, 1)\rangle &= |0, 0\rangle, \leftarrow \ell_0^{0,(1,1)} = 0, \\ E = \frac{5}{4} \rightarrow |0, 0; (1, 2)\rangle &= |0, 1\rangle, \leftarrow \ell_0^{0,(1,2)} = 0, \\ E = \frac{7}{4} \rightarrow \begin{cases} |1, -1; (1, 1)\rangle = \frac{1}{\sqrt{2}}(|0, 2\rangle + i|1, 0\rangle) \leftarrow \ell_{(-1)}^{1,(1,1)} = -\frac{1}{\sqrt{2}} \\ |1, +1; (1, 1)\rangle = \frac{1}{\sqrt{2}}(|0, 2\rangle - i|1, 0\rangle) \leftarrow \ell_{(+1)}^{1,(1,1)} = \frac{1}{\sqrt{2}}, \end{cases} \\ E = \frac{9}{4} \rightarrow \begin{cases} |1, -1; (1, 2)\rangle = \frac{1}{\sqrt{2}}(|0, 3\rangle + i|1, 1\rangle) \leftarrow \ell_{(-1)}^{1,(1,2)} = -\sqrt{\frac{3}{2}} \\ |1, +1; (1, 2)\rangle = \frac{1}{\sqrt{2}}(|0, 3\rangle - i|1, 1\rangle) \leftarrow \ell_{(+1)}^{1,(1,2)} = \sqrt{\frac{3}{2}}, \end{cases} \end{aligned}$$

$$E = \frac{11}{4} \rightarrow \begin{cases} |2, -2; (1, 1)\rangle = \sqrt{\frac{3}{8}}|0, 4\rangle + \frac{i}{\sqrt{2}}|1, 2\rangle - \frac{1}{\sqrt{8}}|2, 0\rangle & \leftarrow \ell_{(-2)}^{2,(1,1)} = -2 \\ |2, 0; (1, 1)\rangle = \frac{1}{2}|0, 4\rangle + \frac{\sqrt{3}}{2}|2, 0\rangle & \leftarrow \ell_{(0)}^{2,(1,1)} = 0 \\ |2, +2; (1, 1)\rangle = \sqrt{\frac{3}{8}}|0, 4\rangle - \frac{i}{\sqrt{2}}|1, 2\rangle - \frac{1}{\sqrt{8}}|2, 0\rangle & \leftarrow \ell_{(+2)}^{2,(1,1)} = +2, \end{cases}$$

$$E = \frac{13}{4} \rightarrow \begin{cases} |2, -2; (1, 2)\rangle = \frac{\sqrt{5}}{4}|0, 5\rangle + \frac{i}{\sqrt{2}}|1, 3\rangle - \frac{\sqrt{3}}{4}|2, 1\rangle & \leftarrow \ell_{(-2)}^{2,(1,2)} = -\sqrt{8} \\ |2, 0; (1, 2)\rangle = \sqrt{\frac{3}{8}}|0, 5\rangle + \sqrt{\frac{3}{8}}|2, 1\rangle & \leftarrow \ell_{(0)}^{2,(1,2)} = 0 \\ |2, +2; (1, 2)\rangle = \frac{\sqrt{5}}{4}|0, 5\rangle - \frac{i}{\sqrt{2}}|1, 3\rangle - \frac{\sqrt{3}}{4}|2, 1\rangle & \leftarrow \ell_{(+2)}^{2,(1,2)} = \sqrt{8}. \end{cases}$$

The following remarks can now be made:

- i) In the basis described by Eqs (15)-(18) it is a trivial matter to distinguish the states belonging to the same irrep for any $m : n$ ratio, while in the Cartesian basis this is true only in the 1:1 case.
- ii) In the 1:2 case we see that the irreps have degeneracies 1, 1, 2, 2, 3, 3, 4, 4, ..., i.e. "two copies" of the $u(2)$ degeneracies 1, 2, 3, 4, ... are obtained.
- iii) In the 1:3 cases the degeneracies are 1, 1, 1, 2, 2, 2, 3, 3, 3, ..., i.e. "three copies" of the $u(2)$ degeneracies are obtained.
- iv) It can be easily seen that the 1:n case corresponds to "n copies" of the $u(2)$ degeneracies.
- v) Cases with both m, n different from unity show more complicated degeneracy patterns, also correctly reproduced by the above formalism. In the 2:3 case, for example, the degeneracy pattern is 1, 1, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 2, 3, 3,
- vi) The only requirement for each energy eigenvalue to correspond to one irrep of the algebra is that m and n have to be mutually prime numbers. If m and n possess a common divisor other than 1, then some energy eigenvalues will correspond to sums of irreps, i.e. to reducible representations.
- vii) The difference between the formalism used here and the one used in ^{54,56,57,60} is that in the latter case for given m and n appropriate operators have to be introduced separately for each set of (p, q) values, while in the present case only one set of operators is introduced.

6.5. Connection to $W_3^{(2)}$

For the special case $m = 1, n = 2$ it should be noticed that the deformed algebra received here coincides with the finite W algebra $W_3^{(2)}$ ⁷³⁻⁷⁶. The commutation relations of the $W_3^{(2)}$ algebra are

$$[H_W, E_W] = 2E_W, \quad [H_W, F_W] = -2F_W, \quad [E_W, F_W] = H_W^2 + C_W,$$

$$[C_W, E_W] = [C_W, F_W] = [C_W, H_W] = 0,$$

while in the $m = 1, n = 2$ case one has the relations

$$[\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \quad [\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \quad [\mathcal{A}, \mathcal{A}^\dagger] = 3S_0^2 - \frac{H^2}{4} - HS_0 + \frac{3}{16},$$

$$[H, \mathcal{A}^\dagger] = [H, \mathcal{A}] = [H, S_0] = 0,$$

with $S_0 = \mathcal{N} + u$ (where u a constant). It is easy to see that the two sets of commutation relations are equivalent by making the identifications

$$F_W = \sigma \mathcal{A}^\dagger, \quad E_W = \rho \mathcal{A}, \quad H_W = -2S_0 + kH, \quad C_W = f(H),$$

with

$$\rho\sigma = \frac{4}{3}, \quad k = \frac{1}{3}, \quad f(H) = -\frac{4}{9}H^2 + \frac{1}{4}.$$

6.6. Summary

In conclusion, the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies equal to m/n , is described dynamically by a deformed version of the $u(2)$ Lie algebra, the order of this algebra being $m+n-1$. The representation modules of this algebra can be generated by using the deformed oscillator algebra. The energy eigenvalues are calculated by the requirement of the existence of finite dimensional representation modules. An “angular momentum” operator useful for labelling degenerate states has also been constructed. The algebras obtained in the special cases with $1:n$ ratios are shown to correspond to generalized parafermionic oscillators. In the special case of $m:n = 1:2$ the resulting algebra has been identified as the finite W algebra $W_3^{(2)}$.

The extension of the present method to the three-dimensional anisotropic quantum harmonic oscillator is already receiving attention, since it is of clear interest in the study of the symmetries underlying the structure of superdeformed and hyperdeformed nuclei.

7. References

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